B.Sc (H) - III Paper - V Gp - A

06-02-24
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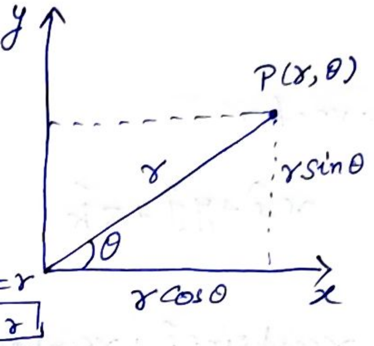
Curvilinear Coordinates continued...

Some orthogonal coordinate system;

① Plane polar coordinates (r,θ)



$x = r\cos\theta,\ y = r\sin\theta$

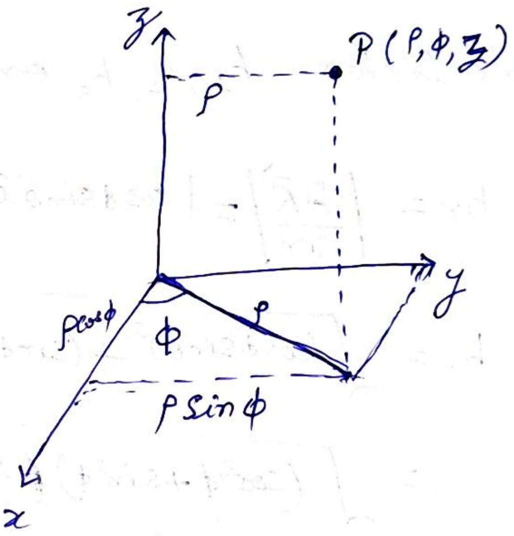
Position vector $\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$

$h_1 = h_r = \left|\frac{\partial\vec{r}}{\partial r}\right| = |\cos\theta\hat{i} + \sin\theta\hat{j}| = 1$

$h_2 = h_\theta = \left|\frac{\partial\vec{r}}{\partial\theta}\right| = |-r\sin\theta\hat{i} + r\cos\theta\hat{j}| = \sqrt{r^2} = r$

$\Rightarrow \boxed{h_r = 1,\ h_\theta = r}$

② Right cylindrical coordinates (ρ, ϕ, z)



$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

Here we can calculate Scaling factors h_1, h_2, h_3 or h_ρ, h_ϕ, h_z

Position vector $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

or $\vec{R} = \rho\cos\phi\hat{i} + \rho\sin\phi\hat{j} + z\hat{k}$

Now $h_1 = h_\rho = \left|\frac{\partial\vec{R}}{\partial\rho}\right| = |\cos\phi\hat{i} + \sin\phi\hat{j} + 0|$

or $h_\rho = |\cos\phi\hat{i} + \sin\phi\hat{j}|$

$= \sqrt{\cos^2\phi + \sin^2\phi}$

$\boxed{h_\rho = 1}$

$h_z = \left|\frac{\partial\vec{R}}{\partial z}\right|$

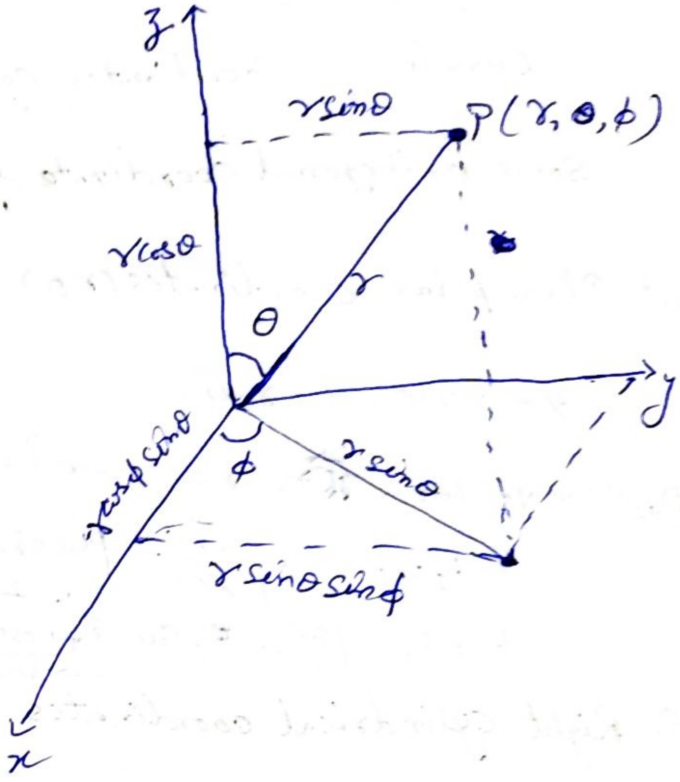
$= |0 + 0 + \hat{k}|$

$\boxed{h_z = 1}$

Similarly $h_2 = h_\phi = \left|\frac{\partial\vec{R}}{\partial\phi}\right| = |-\rho\sin\phi\hat{i} + \rho\cos\phi\hat{j} + 0|$

$\boxed{h_\phi = \sqrt{\rho^2} = \rho}$ $\{\because \cos^2\phi + \sin^2\phi = 1$

③ Spherical Coordinates;



$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

Position vector

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{R} = r\cos\phi\sin\theta\hat{i} + r\sin\phi\sin\theta\hat{j} + r\cos\theta\hat{k}$$

Scale factors h_1, h_2, h_3 or $h_r, h_\theta, h_\phi \Rightarrow$

$$h_r = \left|\frac{\partial\vec{R}}{\partial r}\right| = \left|\cos\phi\sin\theta\hat{i} + \sin\phi\sin\theta\hat{j} + \cos\theta\hat{k}\right|$$

$$h_r = \sqrt{(\cos\phi\sin\theta)^2 + (\sin\phi\sin\theta)^2 + \cos^2\theta}$$

$$= \sqrt{(\cos^2\phi + \sin^2\phi)\sin^2\theta + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$\boxed{h_r = 1}$$

$$h_\theta = \left|\frac{\partial\vec{R}}{\partial\theta}\right| = \left|r\cos\phi\cos\theta\hat{i} + r\sin\phi\cos\theta\hat{j} - r\sin\theta\hat{k}\right|$$

$$h_\theta = \sqrt{r^2\left[(\cos\phi\cos\theta)^2 + (\sin\phi\cos\theta)^2 + \sin^2\theta\right]}$$

$$= r\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}$$

$$= r\sqrt{\cos^2\theta + \sin^2\theta}$$

or $\boxed{h_\theta = r}$

$$h_\phi = \left|\frac{\partial \vec{R}}{\partial \phi}\right|$$

$$= \left|-r\sin\phi\sin\theta\,\hat{i} + r\cos\phi\sin\theta\,\hat{j} + 0\,\hat{k}\right|$$

$$= \sqrt{r^2\sin^2\theta\,(\sin^2\phi + \cos^2\phi)}$$

$$= r\sin\theta\sqrt{\sin^2\phi + \cos^2\phi}$$

or $\boxed{h_\phi = r\sin\theta}$

Thus scale factors for spherical polar coordinates are

$$\boxed{h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta}$$

And volume element is

$$dx\,dy\,dz = h_r h_\theta h_\phi\, dr\, d\theta\, d\phi$$

$$= 1.r.\,r\sin\theta\, dr\,d\theta\,d\phi$$

or $\boxed{dx\,dy\,dz = r^2\sin\theta\, dr\,d\theta\,d\phi}$

Similarly Area element in Plane polar coordinates $\longrightarrow$ $dx\,dy = h_r h_\theta\, dr\, d\theta = 1.r\,dr\,d\theta$

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

$\rightarrow$ Volume element in Right cylindrical coordinates —

$$dx\,dy\,dz = h_\rho h_\phi h_z\, d\rho\, d\phi\, dz$$

$$= 1.\rho.1\, d\rho\, d\phi\, dz$$

or $\boxed{dx\,dy\,dz = \rho\, d\rho\, d\phi\, dz}$